

EXTRAGRADIENT AND LINESEARCH ALGORITHMS FOR SOLVING EQUILIBRIUM PROBLEMS AND FIXED POINT PROBLEMS IN BANACH SPACES

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ABSTRACT. In this paper, using sunny generalized nonexpansive retraction, we propose new extragradient and linesearch algorithms for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of a relatively nonexpansive mapping in Banach spaces. To prove strong convergence of iterates in the extragradient method, we introduce a ϕ -Lipschitz-type condition and assume that the equilibrium bifunction satisfies in this condition. This condition is unnecessary when the linesearch method is used instead of the extragradient method. A numerical example is given to illustrate the usability of our results. Our results generalize, extend and enrich some existing results in the literature.

1. INTRODUCTION

In this paper, we consider the following equilibrium problem (EP) in the sense of Blum and Otteli [5], which consists in finding a point $x^* \in C$ such that

$$f(x^*, y) \geq 0, \quad \forall y \in C,$$

where C is a nonempty, closed and convex subset of a real Banach space E and $f : C \times C \rightarrow \mathbb{R}$ is an equilibrium bifunction, i.e., $f(x, x) = 0$ for all $x \in C$. The solution set of (EP) is denoted by $E(f)$. The equilibrium problem which also known under the name of Ky Fan inequality [12] covers, as special cases, many well-known problems, such as the optimization problem, the variational inequality problem and nonlinear complementarity problem, the saddle point problem, the generalized Nash equilibrium problem in game theory, the fixed point problem and others; (see [22, 28]). Also numerous problems in physic and economic reduce to find a solution of an equilibrium problem. Many methods have been proposed to solve the equilibrium problems see for instance [5, 21, 22, 34, 35]. In 1980, Cohen [10] introduced a useful tool for solving optimization problem which is known as auxiliary problem principle and extended it to variational inequality [11]. In auxiliary problem principle a sequence $\{x_k\}$ is generated as follows: $x_{k+1} \in C$ is a unique solution of the following strongly convex problem

$$\min \left\{ c_k f(x_k, y) + \frac{1}{2} \|x_k - y\| \right\}, \quad (1.1)$$

2010 *Mathematics Subject Classification.* Primary 65K10, 90C25, 47J05, 47J25.

Key words and phrases. Equilibrium problem, Extragradient method, ϕ -Lipschitz-type, Linesearch algorithm, Relatively nonexpansive mapping, Sunny generalized nonexpansive retraction.

where $c_k > 0$. Recently, Mastroeni [20] extended the auxiliary problem principle to equilibrium problems under the assumptions that the equilibrium function f is strongly monotone on $C \times C$ and that f satisfies the following Lipschitz- type condition:

$$f(x, y) + f(y, z) \geq f(x, z) - c_1 \|y - x\|^2 - c_2 \|z - y\|^2, \quad (1.2)$$

for all $x, y, z \in C$ where $c_1, c_2 > 0$. To avoid the monotonicity of f , motivated by Antipin [3], Tran et al. [33] have used an extrapolation step in each iteration after solving (1.1) and suppose that f is pseudomonotone on $C \times C$ which is weaker than monotonicity assumption. They assumed y_k was the unique solution of (1.1) and the unique solution of the following strongly convex problem

$$\min \left\{ c_k f(y_k, y) + \frac{1}{2} \|y - x_k\|^2 \right\},$$

is denoted by $\{x_{k+1}\}$. In special case , when the problem (EP) is a variational inequality problem, this method reduces to the classical extragradient method which has been introduced by Korpelevich [18]. The extragradient method is well known because of its efficiency in numerical tests. In the recent years, many authors obtained extragradient algorithms for solving (EP) in Hilbert spaces where convergence of the proposed algorithms was required f to satisfy a certain Lipschitz-type condition [23, 33, 35]. Lipschitz-type condition depends on two positive parameters c_1 and c_2 which in some cases, they are unknown or difficult to approximate. In other to avoid this requirement, authors used the linesearch technique in a Hilbert space to obtain convergent algorithms for solving equilibrium problem [23, 33, 35].

In this paper, we consider the following auxiliary equilibrium problem $(AUEP)$ for finding $x^* \in C$ such that

$$\rho f(x^*, y) + L(x^*, y) \geq 0, \quad (1.3)$$

for all $y \in C$, where $\rho > 0$ is a regularization parameter and $L : C \times C \rightarrow \mathbb{R}$ be a nonnegative differentiable convex bifunction on C with respect to the second argument y , for each fixed $x \in C$, such that

- (i) $L(x, x) = 0$ for all $x \in C$,
- (ii) $\nabla_2 L(x, x) = 0$ for all $x \in C$.

Where $\nabla_2 L(x, x)$ denotes the gradient of the function $L(x, \cdot)$ at x .

In the recent years, many authors studied the problem of finding a common element of the set of fixed points of a nonlinear mapping and the set of solutions of an equilibrium problem in the framework of Hilbert spaces and Banach spaces, see for instance [7, 25, 29, 31, 35]. In all of these methods, authors have used metric projection in Hilbert spaces and generalized metric projection in Banach spaces.

In this paper, motivated D. Q. Tran et al. [33] and P. T. vuong et al. [35], we introduce new extragradient and linesearch algorithms for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of a relatively nonexpansive

mapping in Banach spaces, by using sunny generalized nonexpansive retraction. Using this method, we prove strong convergence theorems under suitable conditions.

2. PRELIMINARIES

We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}, \quad \forall x \in E.$$

Also, denote the strong convergence and the weak convergence of a sequence $\{x^k\}$ to x in E by $x^k \rightarrow x$ and $x^k \rightharpoonup x$, respectively.

Let $S(E)$ be the unite sphere centered at the origin of E . A Banach space E is strictly convex if $\|\frac{x+y}{2}\| < 1$, whenever $x, y \in S(E)$ and $x \neq y$. Modulus of convexity of E is defined by

$$\delta_E(\epsilon) = \inf\{1 - \frac{1}{2}\|(x+y)\| : \|x\|, \|y\| \leq 1, \|x-y\| \geq \epsilon\}$$

for all $\epsilon \in [0, 2]$. E is said to be uniformly convex if $\delta_E(0) = 0$ and $\delta_E(\epsilon) > 0$ for all $0 < \epsilon \leq 2$. Let p be a fixed real number with $p \geq 2$. A Banach space E is said to be p -uniformly convex [32] if there exists a constant $c > 0$ such that $\delta_E \geq c\epsilon^p$ for all $\epsilon \in [0, 2]$. The Banach space E is called smooth if the limit

$$\lim_{t \rightarrow 0} \frac{\|x+ty\| - \|x\|}{t}, \quad (2.1)$$

exists for all $x, y \in S(E)$. It is also said to be uniformly smooth if the limit (2.1) is attained uniformly for all $x, y \in S(E)$. Every uniformly smooth Banach space E is smooth. If a Banach space E uniformly convex, then E is reflexive and strictly convex [1, 30].

Many properties of the normalized duality mapping J have been given in [1, 30].

We give some of those in the following:

- (1) For every $x \in E$, Jx is nonempty closed convex and bounded subset of E^* .
- (2) If E is smooth or E^* is strictly convex, then J is single-valued.
- (3) If E is strictly convex, then J is one-one.
- (4) If E is reflexive, then J is onto.
- (5) If E is strictly convex, then J is strictly monotone, that is,

$$\langle x - y, Jx - Jy \rangle > 0,$$

for all $x, y \in E$ such that $x \neq y$.

- (6) If E is smooth, strictly convex and reflexive and $J^* : E^* \rightarrow 2^E$ is the normalized duality mapping on E^* , then $J^{-1} = J^*$, $JJ^* = I_{E^*}$ and $J^*J = I_E$, where I_E and I_{E^*} are the identity mapping on E and E^* , respectively.
- (7) If E is uniformly convex and uniformly smooth, then J is uniformly norm-to-norm continuous on bounded sets of E and $J^{-1} = J^*$ is also uniformly norm-to-norm continuous on bounded sets of E^* , i.e., for $\varepsilon > 0$ and $M > 0$, there is a $\delta > 0$ such that

$$\|x\| \leq M, \|y\| \leq M \text{ and } \|x - y\| < \delta \Rightarrow \|Jx - Jy\| < \varepsilon, \quad (2.2)$$

$$\|x^*\| \leq M, \|y^*\| \leq M \text{ and } \|x^* - y^*\| < \delta \Rightarrow \|J^{-1}x^* - J^{-1}y^*\| < \varepsilon. \quad (2.3)$$

Let E be a smooth Banach space, we define the function $\phi : E \times E \rightarrow \mathbb{R}$ by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2,$$

for all $x, y \in E$. Observe that, in a Hilbert space H , $\phi(x, y) = \|x - y\|^2$ for all $x, y \in H$.

It is clear from definition of ϕ that for all $x, y, z, w \in E$,

$$(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2, \quad (2.4)$$

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy \rangle, \quad (2.5)$$

$$2\langle x - y, Jz - Jw \rangle = \phi(x, w) + \phi(y, z) - \phi(x, z) - \phi(y, w). \quad (2.6)$$

If E additionally assumed to be strictly convex, then

$$\phi(x, y) = 0 \iff x = y. \quad (2.7)$$

Also, we define the function $V : E \times E^* \rightarrow \mathbb{R}$ by $V(x, x^*) = \|x\|^2 - 2\langle x, x^* \rangle + \|x^*\|^2$,

for all $x \in E$ and $x^* \in E^*$. That is, $V(x, x^*) = \phi(x, J^{-1}x^*)$ for all $x \in E$ and $x^* \in E^*$.

It is well known that, if E is a reflexive strictly convex and smooth Banach space with E^* as its dual, then

$$V(x, x^*) + 2\langle J^{-1}x^* - x, y^* \rangle \leq V(x, x^* + y^*), \quad (2.8)$$

for all $x \in E$ and all $x^*, y^* \in E^*$ [27].

Let E be a smooth Banach space and C be a nonempty subset of E . A mapping $T : C \rightarrow C$ is called generalized nonexpansive [13] if $F(T) \neq \emptyset$ and

$$\phi(y, Tx) \leq \phi(y, x),$$

for all $x \in C$ and all $y \in F(T)$.

Let C be a closed convex subset of E and $T : C \rightarrow C$ be a mapping. A point p in C is said to be an asymptotic fixed point of T if C contains a sequence $\{x^k\}$ which converges weakly to p such that $\lim_{k \rightarrow \infty} (Tx^k - x^k) = 0$. The set of asymptotic fixed points of T will be denoted by $\hat{F}(T)$. A mapping $T : C \rightarrow C$ is called relatively nonexpansive if $\hat{F}(T) = F(T)$ and $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in C$ and $p \in F(T)$. The asymptotic behavior of relatively nonexpansive mappings was studied in [6]. T is said to be relatively quasi-nonexpansive if $F(T) \neq \emptyset$ and $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in C$ and all $p \in F(T)$. The class of relatively quasi-nonexpansive mapping is broader than the class of relatively nonexpansive mappings which requires $\hat{F}(T) = F(T)$.

It is well known that, if E is a strictly convex and smooth Banach space, C is a nonempty closed convex subset of E and $T : C \rightarrow C$ is a relatively quasi-nonexpansive mapping, then $F(T)$ is a closed convex subset of C [26].

Let D be a nonempty subset of a Banach space E . A mapping $R : E \rightarrow D$ is said to be sunny [13] if

$$R(Rx + t(x - Rx)) = Rx,$$

for all $x \in E$ and all $t \geq 0$. A mapping $R : E \rightarrow D$ is said to be a retraction if $Rx = x$ for all $x \in D$. R is a sunny nonexpansive retraction from E onto D if R is a retraction which is also sunny and nonexpansive. A nonempty subset D of a smooth Banach space E is said to be a generalized nonexpansive retract (resp. sunny generalized nonexpansive retract) of E if there exists a generalized nonexpansive retraction (resp. sunny generalized nonexpansive retraction) R from E onto D .

If E is a smooth, strictly convex and reflexive Banach space, C^* be a nonempty closed convex subset of E^* and Π_{C^*} be the generalized metric projection of E^* onto C^* . Then the $R = J^{-1}\Pi_{C^*}J$ is a sunny generalized nonexpansive retraction of E onto $J^{-1}C^*$ [17].

Remark 2.1. If E is a Hilbert space. Then $R_C = \Pi_C = P_C$.

We need the following lemmas for the proof of our main results.

If C is a convex subset of Banach space E , then we denote by $N_C(\nu)$ the normal cone for C at a point $\nu \in C$, that is

$$N_C(\nu) := \{x^* \in E^* : \langle \nu - y, x^* \rangle \geq 0, \forall y \in C\}.$$

Suppose that E is a Banach space and let $f : E \rightarrow (-\infty, +\infty]$ be a proper function. For $x_0 \in D(f)$, we define the subdifferential of f at x_0 as the subset of E^* given by

$$\partial f(x_0) = \{x^* \in E^* : f(x) \geq f(x_0) + \langle x^*, x - x_0 \rangle, \forall x \in E\}.$$

If $\partial f(x_0) \neq \emptyset$, then we say f is subdifferentiable at x_0 .

Lemma 2.1. *Let C be a nonempty convex subset of a Banach space E and $f : E \rightarrow \mathbb{R}$ be a convex and subdifferentiable function, then f is minimized at $x \in E$ if and only if*

$$0 \in \partial f(x) + N_C(x).$$

Lemma 2.2. [4] *Let E be a reflexive Banach space. If $f : E \rightarrow \mathbb{R} \cup \{+\infty\}$ and $g : E \rightarrow \mathbb{R} \cup \{+\infty\}$ are nontrivial, convex and lower continuous functions and if $0 \in \text{Int}(\text{Dom} f - \text{Dom} g)$, then*

$$\partial(f + g)(x) = \partial f(x) + \partial g(x).$$

Lemma 2.3. [4] *Suppose that a convex function f is continuous on the interior of its domain. Then, for all $x \in \text{Int}(\text{Dom} f)$, $\partial f(x)$ is non-empty and bounded.*

Lemma 2.4. *Let C be a nonempty convex subset of a Banach space E and let $f : C \times C \rightarrow \mathbb{R}$ be an equilibrium bifunction and convex respect to the second variable. then the following statements are equivalent:*

- (i) x^* is a solution to $E(f)$,
- (ii) x^* is a solution to the problem

$$\min_{y \in C} f(x^*, y).$$

Proof. Using Lemma 2.1, we get desired results. □

Equivalence between $E(f)$ and $(AUEP)$ is stated in the following lemma.

Lemma 2.5. *Let C nonempty, convex and closed subset of a reflexive Banach space E and $f : C \times C \rightarrow \mathbb{R}$ be an equilibrium bifunction and let $x^* \in C$. suppose that $f(x^*, \cdot) : C \rightarrow \mathbb{R}$ is convex and subdifferentiable on C . Let $L : C \times C \rightarrow \mathbb{R}_+$ be a differentiable convex function on C with respect to the second argument y such that*

- (i) $L(x^*, x^*) = 0$,
- (ii) $\nabla_2 L(x^*, x^*) = 0$.

Then $x^ \in C$ is a solution to $E(f)$ if and only if x^* is a solution to $(AUEP)$.*

Proof. It is clear from lemmas 2.2 and 2.4. □

Lemma 2.6. [13] *Let C be a nonempty closed sunny generalized nonexpansive retract of a smooth and strictly convex Banach space E . Then the sunny generalized nonexpansive retraction from E onto C is uniquely determined.*

Lemma 2.7. [13] *Let C be a nonempty closed subset of a smooth and strictly convex Banach space E such that there exists a sunny generalized nonexpansive retraction R from E onto C and let $(x, z) \in E \times C$. Then the following hold:*

- (1) $z = Rx$ if and only if $\langle x - z, Jy - Jz \rangle \leq 0$ for all $y \in C$,
- (2) $\phi(z, Rx) + \phi(Rx, x) \leq \phi(z, x)$.

Lemma 2.8. [17] *Let E be a smooth, strictly convex and reflexive Banach space and C be a nonempty closed sunny generalized nonexpansive retract of E . Let R be the sunny generalized nonexpansive retraction from E onto C and $(x, z) \in E \times C$. Then the following are equivalent:*

- (1) $z = Rx$,
- (2) $\phi(z, x) = \min_{y \in C} \phi(y, x)$

Lemma 2.9. [13] *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty closed subset of E . Then the following are equivalent:*

- (1) C is a sunny generalized nonexpansive retract of E ,
- (2) C is a generalized retract of E ,
- (3) JC is closed and convex.

Lemma 2.10. [36] *Let E be a 2-uniformly convex and smooth Banach space. Then, for all $x, y \in E$, we have*

$$\|x - y\| \leq \frac{2}{c^2} \|Jx - Jy\|,$$

where $\frac{1}{c}$ ($0 \leq c \leq 1$) is the 2-uniformly convex constant of E .

Lemma 2.11. [9] *Suppose $p > 1$ is a real number, then the following are equivalent*

- (i) E is a p -uniformly convex Banach space,

(ii) there exists $\tau > 0$, such that for each $f_x \in J_p(x)$ and $f_y \in J_p(y)$, we have

$$\langle x - y, f_x - f_y \rangle \geq \tau \|x - y\|^p.$$

Where $J_p(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\| \|x^*\|, \|x^*\| = \|x\|^{p-1}\}$.

Lemma 2.12. [15] *Let E be a uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, 2r] \rightarrow [0, \infty)$, $g(0) = 0$ such that*

$$g(\|x - y\|) \leq \phi(x, y),$$

for all $x, y \in B_r(0) = \{z \in E : \|z\| \leq r\}$.

Lemma 2.13. [8] *Let E be a uniformly convex Banach space. Then there exists a continuous strictly increasing convex function $g : [0, 2r] \rightarrow [0, \infty)$, $g(0) = 0$ such that*

$$\|\lambda x + \mu y + \gamma z\|^2 \leq \lambda \|x\|^2 + \mu \|y\|^2 + \gamma \|z\|^2 - \lambda \mu g(\|x - y\|),$$

for all $x, y, z \in B_r(0) = \{z \in E : \|z\| \leq r\}$ and all $\lambda, \mu, \gamma \in [0, 1]$ with $\lambda + \mu + \gamma = 1$.

Lemma 2.14. [15] *Let E be a uniformly convex and smooth Banach space and let $\{x^k\}$ and $\{y^k\}$ be two sequences of E . If $\phi(x^k, y^k) \rightarrow 0$ and either $\{x^k\}$ or $\{y^k\}$ is bounded, then $x^k - y^k \rightarrow 0$.*

Lemma 2.15. [16] *Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two positive and bounded sequences in \mathbb{R} , then*

$$\begin{aligned} (\liminf_{n \rightarrow \infty} \alpha_n) \times (\liminf_{n \rightarrow \infty} \beta_n) &\leq \liminf_{n \rightarrow \infty} (\alpha_n \beta_n) \\ &\leq (\liminf_{n \rightarrow \infty} \alpha_n) \times (\limsup_{n \rightarrow \infty} \beta_n) \\ &\leq \limsup_{n \rightarrow \infty} (\alpha_n \beta_n) \\ &\leq (\limsup_{n \rightarrow \infty} \alpha_n) \times (\limsup_{n \rightarrow \infty} \beta_n). \end{aligned}$$

3. AN EXTRAGRADIENT ALGORITHM

In this section, we present an algorithm for finding a solution of the (EP) which is also the common element of the set of fixed points of a relatively nonexpansive mapping.

Here, we assume that bifunction $f : C \times C \rightarrow \mathbb{R}$ satisfies in following conditions which C is nonempty, convex and closed subset of uniformly convex and uniformly smooth Banach space E ,

- (A1) $f(x, x) = 0$ for all $x \in C$,
- (A2) f is pseudomonotone on C , i.e., $f(x, y) \geq 0 \implies f(y, x) \leq 0$ for all $x, y \in C$,
- (A3) f is jointly weakly continuous on $C \times C$, i.e., if $x, y \in C$ and $\{x_n\}$ and $\{y_n\}$ are two sequences in C converging weakly to x and y , respectively, then $f(x_n, y_n) \rightarrow f(x, y)$,
- (A4) $f(x, \cdot)$ is convex, lower semicontinuous and subdifferentiable on C for every $x \in C$,
- (A5) f satisfies ϕ -Lipschitz-type condition: $\exists c_1 > 0, \exists c_2 > 0$, such that for every $x, y, z \in C$

$$f(x, y) + f(y, z) \geq f(x, z) - c_1 \phi(y, x) - c_2 \phi(z, y). \quad (3.1)$$

It is easy to see that if f satisfies the properties $(A_1) - (A_4)$, then the set $E(f)$ of solutions of an equilibrium problem is closed and convex. Indeed, when E is a Hilbert space, ϕ -Lipschitz-type condition reduces to Lipschitz-type condition (1.2).

Throughout the paper S is a relatively nonexpansive self-mapping of C .

Algorithm 1

Step 0.: Suppose that $\{\alpha_n\} \subseteq [a, e]$ for some $0 < a < e < 1$, $\{\beta_n\} \subseteq [d, b]$ for some $0 < d < b < 1$ and $\{\lambda_n\} \subseteq (0, 1]$ such that $\{\lambda_n\} \subseteq [\lambda_{min}, \lambda_{max}]$, where

$$0 < \lambda_{min} \leq \lambda_{max} < \min\left\{\frac{1}{2c_1}, \frac{1}{2c_2}\right\}.$$

Step 1.: Let $x_0 \in C$. Set $n=0$.

Step 2.: Compute y_n and x_n , such that

$$y_n = \arg \min_{y \in C} \left\{ \lambda_n f(x_n, y) + \frac{1}{2} \phi(y, x_n) \right\},$$

$$z_n = \arg \min_{y \in C} \left\{ \lambda_n f(y_n, y) + \frac{1}{2} \phi(y, x_n) \right\}.$$

Step 3.: Compute $t_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)(\beta_n Jz_n + (1 - \beta_n)JSz_n))$.

If $y_n = x_n$ and $t_n = x_n$, then $x_n \in E(f) \cap F(S)$ and go to step 4.

Step 4.: Compute $x_{n+1} = R_{C_n \cap D_n} x_0$, where $R_{C_n \cap D_n}$ is the sunny generalized nonexpansive retraction from E onto $C_n \cap D_n$ and

$$C_n = \{z \in C : \phi(z, t_n) \leq \phi(z, x_n)\},$$

$$D_n = \{z \in C : \langle Jx_n - Jz, x_0 - x_n \rangle \geq 0\}.$$

Step 5.: set $n := n + 1$ and go to Step 2.

Before proving the strong convergence of the iterates generated by Algorithm 1, we prove following lemmas.

Lemma 3.1. *For every $x^* \in E(f)$ and $n \in \mathbb{N}$, we obtain*

- (i) $\langle Jx_n - Jy_n, y - y_n \rangle \leq \lambda_n f(x_n, y) - \lambda_n f(x_n, y_n), \quad \forall y \in C,$
- (ii) $\phi(x^*, z_n) \leq \phi(x^*, x_n) - (1 - 2\lambda_n c_1)\phi(y_n, x_n) - (1 - 2\lambda_n c_2)\phi(z_n, y_n).$

Proof. By the condition (A4) for $f(x, \cdot)$ and from lemmas 2.1 and 2.2, we obtain

$$z_n = \arg \min_{y \in C} \left\{ \lambda_n f(y_n, y) + \frac{1}{2} \phi(y, x_n) \right\}$$

if and only if

$$0 \in \lambda_n \partial_2 f(y_n, z_n) + \frac{1}{2} \nabla_1 \phi(z_n, x_n) + N_C(z_n).$$

This implies that $w \in \partial_2 f(y_n, z_n)$ and $\bar{w} \in N_C(z_n)$ exist such that

$$0 = \lambda_n w + Jz_n - Jx_n + \bar{w}, \tag{3.2}$$

so, from definition of $\partial_2 f(y_n, z_n)$, we obtain

$$\langle w, y - z_n \rangle \leq f(y_n, y) - f(y_n, z_n),$$

for all $y \in C$. Set $y = x^*$, we have

$$\langle w, x^* - z_n \rangle \leq f(y_n, x^*) - f(y_n, z_n).$$

So, by definition of the $N_C(z_n)$ and equality (3.2), we get

$$\lambda_n \langle w, z_n - y \rangle \leq \langle Jz_n - Jx_n, y - z_n \rangle, \quad (3.3)$$

for all $y \in C$. Put $y = x^*$ in inequality 3.3, we have

$$\langle Jz_n - Jx_n, x^* - z_n \rangle \geq \lambda_n \{f(y_n, z_n) - f(y_n, x^*)\} \geq \lambda_n f(y_n, z_n), \quad (3.4)$$

since $f(x^*, y_n) \geq 0$ and f is pseudomonotone on C . Replacing x, y and z by x_n, y_n and z_n in inequality (3.1), respectively, we get

$$f(y_n, z_n) \geq f(x_n, z_n) - f(x_n, y_n) - c_1 \phi(y_n, x_n) - c_2 \phi(z_n, y_n). \quad (3.5)$$

In a similar way, since $y_n = \arg \min_{y \in C} \{\lambda_n f(x_n, y) + \frac{1}{2} \phi(y, x_n)\}$, we have

$$\langle Jx_n - Jy_n, y - y_n \rangle \leq \lambda_n \{f(x_n, y) - f(x_n, y_n)\},$$

for all $y \in C$, hence (i) is proved. Let $y = z_n$ in above inequality, we obtain

$$\langle Jx_n - Jy_n, z_n - y_n \rangle \leq \lambda_n \{f(x_n, z_n) - f(x_n, y_n)\}. \quad (3.6)$$

Combining inequalities (3.4), (3.5) and (3.6), we get

$$2\langle Jz_n - Jx_n, x^* - z_n \rangle \geq 2\langle Jy_n - Jx_n, y_n - z_n \rangle - 2\lambda_n c_1 \phi(y_n, x_n) - 2\lambda_n c_2 \phi(z_n, y_n). \quad (3.7)$$

From inequality (3.7) and (2.5), we have

$$\phi(x^*, x_n) - \phi(x^*, z_n) \geq \phi(z_n, y_n) + \phi(y_n, x_n) - 2\lambda_n c_1 \phi(y_n, x_n) - 2\lambda_n c_2 \phi(z_n, y_n).$$

Hence, (ii) is proved. \square

Remark 3.1. In a real Hilbert space E , Lemma 3.1 is reduced to Lemma 3.1 in [2].

Lemma 3.2. In Algorithm 1, we obtain the unique optimal solutions y_n and z_n .

Proof. Let $y_n, \acute{y}_n \in \arg \min_{y \in C} \{\lambda_n f(x_n, y) + \frac{1}{2} \phi(y, x_n)\}$, then using Lemma 3.1(i), we have

$$\langle Jx_n - Jy_n, y - y_n \rangle \leq \lambda_n f(x_n, y) - \lambda_n f(x_n, y_n), \quad \forall y \in C, \quad (3.8)$$

$$\langle Jx_n - J\acute{y}_n, y - \acute{y}_n \rangle \leq \lambda_n f(x_n, y) - \lambda_n f(x_n, \acute{y}_n), \quad \forall y \in C. \quad (3.9)$$

Set $y = \acute{y}_n$ in inequality (3.8) and $y = y_n$ in inequality (3.9). Hence, we get

$$\langle J\acute{y}_n - Jy_n, \acute{y}_n - y_n \rangle \leq 0.$$

Since J is monotone and one-one, we obtain $y_n = \acute{y}_n$. In a similar way, also z_n is unique. \square

Remark 3.2. If E is a real Hilbert space, then Algorithm 1 is the same Extragradient Algorithm in [35] provided that the sequence $\{\alpha_n\}$ satisfies the conditions of Step 0 of Algorithm 1.

Lemma 3.3. *For every $x^* \in E(f) \cap F(S)$ and $n \in \mathbb{N}$, we get*

$$\phi(x^*, t_n) \leq \phi(x^*, x_n).$$

Proof. From Lemma 3.1, by the convexity of $\|\cdot\|^2$ and by the definition of functions ϕ and S , we have

$$\begin{aligned} \phi(x^*, t_n) &= \phi(x^*, J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)(\beta_n Jz_n + (1 - \beta_n)JSz_n))) \\ &= \|x^*\|^2 + \|\alpha_n Jx_n + (1 - \alpha_n)(\beta_n Jz_n + (1 - \beta_n)JSz_n)\| \\ &\quad - 2\langle x^*, \alpha_n Jx_n + (1 - \alpha_n)(\beta_n Jz_n + (1 - \beta_n)JSz_n) \rangle \\ &\leq \|x^*\|^2 + \alpha_n \|x_n\|^2 + (1 - \alpha_n)\beta_n \|z_n\|^2 + (1 - \alpha_n)(1 - \beta_n)\|Sz_n\|^2 \\ &\quad - 2\alpha_n \langle x^*, Jx_n \rangle - 2(1 - \alpha_n)\beta_n \langle x^*, Jz_n \rangle - 2(1 - \alpha_n)(1 - \beta_n) \langle x^*, JSz_n \rangle \\ &= \alpha_n \phi(x^*, x_n) + (1 - \alpha_n)\beta_n \phi(x^*, z_n) + (1 - \alpha_n)(1 - \beta_n) \phi(x^*, Sz_n) \\ &\leq \phi(x^*, x_n). \end{aligned} \tag{3.10}$$

□

We examine the stopping condition in the following lemma.

Lemma 3.4. *Let $y_n = x_n$, then $x_n \in E(f)$. If $y_n = x_n$ and $t_n = x_n$, then $x_n \in E(f) \cap F(S)$.*

Proof. Suppose $y_n = x_n$, then by the definition of y_n , condition (A1), property of ϕ (2.7) and since $0 < \lambda_{\min} \leq \lambda_n \leq \lambda_{\max} \leq 1$, we have

$$0 \leq \lambda_n f(x_n, y_n) + \frac{1}{2} \phi(y_n, x_n) \leq f(x_n, y) + \frac{1}{2} \phi(y, x_n),$$

for all $y \in C$. Set $\phi(y, x_n) = L(x_n, y)$, hence Lemma 2.5 implies that $x_n \in E(f)$.

Let $y_n = x_n$ and $t_n = x_n$, we have that $z_n = x_n$ and since J^{-1} is one-one, we get

$$Jx_n = \alpha_n Jx_n + (1 - \alpha_n)(\beta_n Jx_n + (1 - \beta_n)JSx_n).$$

Since $1 - \alpha_n > 0$ and $1 - \beta_n > 0$, it follows that $Jx_n = JSx_n$ and since J is one-one, we get $x_n = Sx_n$. So $x_n \in F(S)$ □

Remark 3.3. In a real Hilbert space E , Lemma 3.4 is the same Proposition 3.5 in [35] with different proof, provided that the sequence $\{\alpha_n\}$ satisfies the conditions of Step 0 of Algorithm 1.

Theorem 3.1. *Let C be a nonempty closed convex subset of a uniformly convex, uniformly smooth Banach space E . Assume that $f : C \times C \rightarrow \mathbb{R}$ is a bifunction which satisfies conditions (A1) – (A5) and $S : C \rightarrow C$ is a relatively nonexpansive mapping such that*

$$\Omega := E(f) \cap F(S) \neq \emptyset.$$

Then sequences $\{x_n\}_{n=1}^\infty$, $\{y_n\}_{n=1}^\infty$, $\{z_n\}_{n=1}^\infty$ and $\{t_n\}_{n=1}^\infty$ generated by Algorithm 1 converge strongly to the some solution $u^ \in \Omega$, where $u^* = R_\Omega x_0$, and R_Ω is sunny generalized nonexpansive retraction from E onto Ω .*

Proof. At First, using induction we show that $\Omega \subseteq C_n \cap D_n$ for all $n \geq 0$. Let $x^* \in \Omega$, from Lemma 3.3, we get $\Omega \subseteq C_n$ for all $n \geq 0$. Now, we show that $\Omega \subseteq D_n$ for all $n \geq 0$. It is clear that $\Omega \subseteq D_0$. Suppose that $\Omega \subseteq D_n$, i.e. $\langle Jx_n - Jx^*, x_0 - x_n \rangle \geq 0$, for all $x^* \in \Omega$. Since $x_{n+1} = R_{C_n \cap D_n} x_0$, using Lemma 2.7, we get $\langle Jx_{n+1} - Jz, x_0 - x_{n+1} \rangle \geq 0$, for all $z \in C_n \cap D_n$. This implies that $x^* \in D_{n+1}$. Therefore $\Omega \subseteq D_{n+1}$.

Let $x^* \in \Omega \subseteq D_{n+1}$. Since $x_{n+1} \in D_n$, using successively equality (2.5), it is easy to see that the $\{\phi(x_0, x_n)\}$ is increasing and bounded from above by $\phi(x_0, x^*)$, so $\lim_{n \rightarrow \infty} \phi(x_0, x_n)$ exists. This yields that $\{\phi(x_0, x_n)\}$ is bounded. From inequality (2.4), we know that $\{x_n\}$ is bounded. It is clear that $\lim_{n \rightarrow \infty} \phi(x_n, x_{n+1}) = 0$, so Lemma 2.14 implies that $\lim_{n \rightarrow \infty} \|x_n - x_{n+1}\| = 0$ and therefore $\{x_n\}$ converges strongly to $\bar{x} \in C$. Since J is uniformly norm-to-norm continuous on bounded sets, from equality (2.6), we obtain that $\lim_{n \rightarrow \infty} \phi(x_{n+1}, x_n) = 0$, and $\lim_{n \rightarrow \infty} \phi(\nu, x_n)$ exists for all $\nu \in C$. Since $x_{n+1} \in C_n$, we have $\lim_{n \rightarrow \infty} \phi(x_{n+1}, t_n) = 0$ and from Lemma 2.14, we deduce that $\lim_{n \rightarrow \infty} \|x_{n+1} - t_n\| = 0$, thus $\lim_{n \rightarrow \infty} \|x_n - t_n\| = 0$ which implies that $\{t_n\}$ converges strongly to \bar{x} . Using norm-to-norm continuity of J on bounded sets, we conclude that $\lim_{n \rightarrow \infty} \|Jx_n - Jt_n\| = 0$ and therefore

$$\lim_{n \rightarrow \infty} \phi(x^*, x_n) = \lim_{n \rightarrow \infty} \phi(x^*, t_n). \quad (3.11)$$

Using Lemma 3.1 (ii), we obtain $\phi(x^*, z_n) \leq \phi(x^*, x_n)$. From inequality (2.4) and the definition of S , we derive that $\{z_n\}$ and $\{Sz_n\}$ are bounded. Let $r_1 = \sup_{n \geq 0} \{\|x_n\|, \|z_n\|\}$ and $r_2 = \sup_{n \geq 0} \{\|z_n\|, \|Sz_n\|\}$. So, by Lemma 2.13, there exists a continuous, strictly increasing and convex function $g_1 : [0, 2r_1] \rightarrow \mathbb{R}$ with $g_1(0) = 0$ such that for all $x^* \in \Omega$, we get

$$\begin{aligned} \phi(x^*, t_n) &= \phi(x^*, J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)(\beta_n Jz_n + (1 - \beta_n)JSz_n))) \\ &\leq \|x^*\|^2 + \alpha_n \|x_n\|^2 + (1 - \alpha_n)\beta_n \|z_n\|^2 + (1 - \alpha_n)(1 - \beta_n) \|Sz_n\|^2 \\ &\quad - 2\alpha_n \langle x^*, Jx_n \rangle - 2(1 - \alpha_n)\beta_n \langle x^*, Jz_n \rangle - 2(1 - \alpha_n)(1 - \beta_n) \langle x^*, JSz_n \rangle \\ &\quad - \alpha_n(1 - \alpha_n)\beta_n g_1(\|Jz_n - Jx_n\|) \\ &\leq \phi(x^*, x_n) - \alpha_n(1 - \alpha_n)\beta_n g_1(\|Jz_n - Jx_n\|), \end{aligned} \quad (3.12)$$

and using the same argument, there exists a continuous, strictly increasing and convex function $g_2 : [0, 2r_2] \rightarrow \mathbb{R}$ with $g_2(0) = 0$ such that for all $x^* \in \Omega$, we have

$$\phi(x^*, t_n) \leq \phi(x^*, x_n) - (1 - \alpha_n)^2 \beta_n (1 - \beta_n) g_2(\|Jz_n - JSz_n\|) = 0,$$

which imply

$$\alpha_n(1 - \alpha_n)\beta_n g_1(\|Jz_n - Jx_n\|) \leq \phi(x^*, x_n) - \phi(x^*, t_n), \quad (3.13)$$

$$(1 - \alpha_n)^2 \beta_n (1 - \beta_n) g_2(\|Jz_n - JSz_n\|) \leq \phi(x^*, x_n) - \phi(x^*, t_n). \quad (3.14)$$

By letting $n \rightarrow \infty$ in inequalities (3.13) and (3.14), using Lemma 2.15 and equality (3.11), we obtain

$$\lim_{n \rightarrow \infty} g_1(\|Jz_n - Jx_n\|) = 0 \quad \& \quad \lim_{n \rightarrow \infty} g_2(\|Jz_n - JSz_n\|) = 0.$$

From the properties of g_1 and g_2 , we have

$$\lim_{n \rightarrow \infty} \|Jz_n - Jx_n\| = 0 \quad \& \quad \lim_{n \rightarrow \infty} \|Jz_n - JSz_n\| = 0. \quad (3.15)$$

So from (2.3), we obtain

$$\lim_{n \rightarrow \infty} \|z_n - x_n\| = 0 \quad \& \quad \lim_{n \rightarrow \infty} \|z_n - Sz_n\| = 0, \quad (3.16)$$

since J^{-1} is uniformly norm-to-norm continuous on bounded sets. By the same reason as in the proof of (3.11), we can conclude from (3.15) and (3.16) that

$$\lim_{n \rightarrow \infty} \phi(x^*, x_n) = \lim_{n \rightarrow \infty} \phi(x^*, z_n), \quad (3.17)$$

for all $x^* \in \Omega$. Using Lemma 3.1 (ii), we have

$$(1 - 2\lambda_n c_1)\phi(y_n, x_n) + (1 - 2\lambda_n c_2)\phi(z_n, y_n) \leq \phi(x^*, x_n) - \phi(x^*, z_n). \quad (3.18)$$

for all $x^* \in \Omega$. Taking the limits as $n \rightarrow \infty$ in inequality (3.18), using equality (3.17), we get

$$\lim_{n \rightarrow \infty} \phi(y_n, x_n) = 0 \quad \& \quad \lim_{n \rightarrow \infty} \phi(z_n, y_n) = 0. \quad (3.19)$$

Since $\{x_n\}$ and $\{z_n\}$ are bounded, it follows from Lemma 2.14 that

$$\lim_{n \rightarrow \infty} \|y_n - x_n\| = 0 \quad \& \quad \lim_{n \rightarrow \infty} \|z_n - y_n\| = 0,$$

which imply $\{y_n\}$ and $\{z_n\}$ converges strongly to $\bar{x} \in C$.

Now, we prove that $\bar{x} \in E(f)$. It follows from the definition of y_n that

$$\lambda_n f(x_n, y_n) + \frac{1}{2}\phi(y_n, x_n) \leq \lambda_n f(x_n, y) + \frac{1}{2}\phi(y, x_n) \quad (3.20)$$

for all $y \in C$. By letting $n \rightarrow \infty$ in inequality (3.20), it follows from equality (3.19), conditions (A1) and (A3) and uniformly norm-to-norm continuity of J on bounded sets that

$$0 \leq f(\bar{x}, y) + \phi(y, \bar{x}),$$

because of $0 < \lambda_{\min} \leq \lambda_n \leq \lambda_{\max} \leq 1$. Hence, letting $\phi(y, \bar{x}) = L(\bar{x}, y)$, Lemma 2.5 implies that $\bar{x} \in E(f)$.

Now, since $z_n \rightarrow \bar{x}$, from (3.16), we get $\bar{x} \in \hat{F}(S)$. So, using the definition of S , we have $\bar{x} \in F(S)$. Setting $z = u^*$ in Lemma 2.7, since $x_{n+1} = R_{C_n \cap D_n} x_0$ and ϕ is continuous respect to the first argument, we obtain

$$\phi(u^*, x_0) \geq \lim_{n \rightarrow \infty} \phi(x_{n+1}, x_0) = \lim_{n \rightarrow \infty} \phi(x_n, x_0) = \phi(\bar{x}, x_0),$$

also, using Lemma 2.8, we have

$$\phi(u^*, x_0) \leq \phi(y, x_0),$$

for all $y \in \Omega$, because of $u^* = R_\Omega x_0$. Therefore $\bar{x} = u^*$ and consequently the sequences $\{x_n\}_{n=1}^\infty$, $\{y_n\}_{n=1}^\infty$, $\{z_n\}_{n=1}^\infty$ and $\{t_n\}_{n=1}^\infty$ converge strongly to $R_\Omega x_0$. \square

Remark 3.4. If E is a real Hilbert space, then Theorem 3.1 is the same Theorem 3.1 in [35] for a nonexpansive mapping S with different proof, provided that the sequence $\{\alpha_n\}$ satisfies the conditions of Step 0 of Algorithm 1.

4. A LINESEARCH ALGORITHM

As we see in the previous section, ϕ -Lipschitz-type condition (A5) depends on two positive parameters c_1 and c_2 . In some cases, these parameters are unknown or difficult to approximate. To avoid this difficulty, using linesearch method, we modify Extragradient Algorithm. We prove strong convergence of this new algorithm without assuming the ϕ -Lipschitz-type condition. linesearch method has a good efficiency in numerical tests.

Here, we assume that bifunction $f : \Delta \times \Delta \rightarrow \mathbb{R}$ satisfies in conditions (A1), (A2) and (A4) and also in following condition which C is nonempty, convex and closed of 2-uniformly convex, uniformly smooth Banach space E and Δ is an open convex set containing C , (A3*) f is jointly weakly continuous on $\Delta \times \Delta$, i.e., if $x, y \in C$ and $\{x_n\}$ and $\{y_n\}$ are two sequences in Δ converging weakly to x and y , respectively, then $f(x_n, y_n) \rightarrow f(x, y)$.

Algorithm 2

Step 0.: Let $\alpha \in (0, 1)$, $\gamma \in (0, 1)$ and suppose that $\{\alpha_n\} \subseteq [a, e]$ for some $0 < a < e < 1$, $\{\beta_n\} \subseteq [d, b]$ for some $0 < d < b < 1$, $\{\lambda_n\} \subseteq [\lambda, 1]$, where $0 < \lambda \leq 1$ and $0 < \nu < \frac{c^2}{2}$, where $\frac{1}{c}$ ($0 < c \leq 1$) is the 2-uniformly convexity constant of E .

Step 1.: Let $x_0 \in C$. set $n = 0$.

Step 2.: Obtain the unique optimal solution y_n by Solving the following convex problem

$$\min_{y \in C} \{ \lambda_n f(x_n, y) + \frac{1}{2} \phi(x_n, y) \} \quad (4.1)$$

Step 3.: If $y_n = x_n$, then set $z_n = x_n$. Otherwise

Step 3.1.: Find m the smallest nonnegative integer such that

$$\begin{cases} f(z_{n,m}, x_n) - f(z_{n,m}, y_n) \geq \frac{\alpha}{2\lambda_n} \phi(y_n, x_n) \text{ where} \\ z_{n,m} = (1 - \gamma^m)x_n + \gamma^m y_n. \end{cases} \quad (4.2)$$

Step 3.2.: Set $\rho_n = \gamma^m$, $z_n = z_{n,m}$ and go to Step 4.

Step 4.: Choose $g_n \in \partial_2 f(z_n, x_n)$ and compute $w_n = R_C J^{-1}(Jx_n - \sigma_n g_n)$. If $y_n \neq x_n$, then $\sigma_n = \frac{\nu f(z_n, x_n)}{\|g_n\|^2}$ and $\sigma_n = 0$ otherwise.

Step 5.: Compute $t_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)(\beta_n Jw_n + (1 - \beta_n)JSw_n))$. If $y_n = x_n$ and $t_n = x_n$, then STOP: $x_n \in E(f) \cap F(S)$. Otherwise, go to Step 6.

Step 6.: Compute $x_{n+1} = R_{C_n \cap D_n} x_0$, where

$$C_n = \{z \in C : \phi(z, t_n) \leq \phi(z, x_n)\},$$

$$D_n = \{z \in C : \langle Jx_n - Jz, x_0 - x_n \rangle \geq 0\}.$$

Step 7.: Set $n := n+1$, and go to Step 2.

The following lemma shows that linesearch corresponding to x_n and y_n (Step 3.1) is well defined.

Lemma 4.1. *Assume that $y_n = x_n$ for some $n \in \mathbb{N}$. Then*

- (i) *There exists a nonnegative integer m such that the inequality in (4.2) holds.*
- (ii) $f(z_n, x_n) > 0$.
- (iii) $0 \notin \partial_2 f(z_n, x_n)$.

Proof. Suppose that $n \in \mathbb{N}$. Assume towards a contradiction that for each nonnegative integer m ,

$$f(z_{n,m}, x_n) - f(z_{n,m}, y_n) < \frac{\alpha}{2\lambda_n} \phi(y_n, x_n), \quad (4.3)$$

where $z_{n,m} = (1 - \gamma^m)x_n + \gamma^m y_n$. It is easy to see that $z_{n,m} \rightarrow x_n$ as $m \rightarrow \infty$. Using condition (A3*), we obtain

$$f(z_{n,m}, x_n) \rightarrow f(x_n, x_n) \quad \& \quad f(z_{n,m}, y_n) \rightarrow f(x_n, y_n). \quad (4.4)$$

Since $f(x_n, x_n) = 0$, letting $m \rightarrow \infty$ in inequality (4.3), we get

$$0 \leq f(x_n, y_n) + \frac{\alpha}{2\lambda_n} \phi(y_n, x_n). \quad (4.5)$$

Because of y_n is a solution of the convex optimization problem (4.1), we deduce

$$\lambda_n f(x_n, y) + \frac{1}{2} \phi(y, x_n) \geq \lambda_n f(x_n, y_n) + \frac{1}{2} \phi(y_n, x_n),$$

for all $y \in C$. If $y = x_n$, then

$$\lambda_n f(x_n, y_n) + \frac{1}{2} \phi(y_n, x_n) \leq 0. \quad (4.6)$$

It follows from inequalities (4.5) and (4.6) that

$$f(x_n, y_n) + \frac{1}{2\lambda_n} \phi(y_n, x_n) \leq f(x_n, y_n) + \frac{\alpha}{2\lambda_n} \phi(y_n, x_n). \quad (4.7)$$

Therefore from inequality (4.7), we obtain

$$\frac{1 - \alpha}{2} \phi(y_n, x_n) \leq 0,$$

since $\lambda_n \leq 1$. It follows from (2.7) that $\phi(y_n, x_n) > 0$, because of $y_n \neq x_n$. Thus, $1 - \alpha \leq 0$ or $\alpha \geq 1$ where contradict the assumption $\alpha \in (0, 1)$. So, (i) is proved.

Now, we prove (ii). Since f is convex, we obtain

$$\rho_n f(z_n, y_n) + (1 - \rho_n) f(z_n, x_n) \geq f(z_n, z_n) = 0. \quad (4.8)$$

consequently from inequality (4.8), we get

$$f(z_n, x_n) \geq \rho_n [f(z_n, x_n) - f(z_n, y_n)] \geq \frac{\alpha \rho_n}{\lambda_n} \phi(y_n, x_n) > 0,$$

because of $y_n \neq x_n$. Therefore $f(z_n, x_n) > 0$.

The proof (iii) can be found in [33] (Lemma 4.5). □

Remark 4.1. If E is a real Hilbert space, then Lemma 4.1 is reduced to Proposition 4.1 in [35] when $\alpha \in (0, 1)$.

We examine the stopping condition in the following lemma where its proof is similar to the proof Lemma 3.4.

Lemma 4.2. *Let $y_n = x_n$, then $x_n \in E(f)$. If $y_n = x_n$ and $t_n = x_n$, then $w_n = x_n$ and $x_n \in E(f) \cap F(S)$.*

Remark 4.2. If E is a real Hilbert space, then Lemma 4.2 is the same Proposition 4.2 in [35] with different proof, provided that the sequence $\{\alpha_n\}$ satisfies the conditions of Step 0 of Algorithm 2.

Lemma 4.3. *Suppose that $f : \Delta \times \Delta \rightarrow \mathbb{R}$ is a bifunction satisfying conditions (A3*) and (A4). Let $\{x_n\}$ and $\{z_n\}$ be two sequences in Δ such that Suppose $x_n \rightarrow \bar{x}$ and $z_n \rightarrow \bar{z}$, where $\bar{x}, \bar{z} \in \Delta$. Then we have*

$$\partial_2 f(z_n, x_n) \subseteq \partial_2 f(\bar{z}, \bar{x}).$$

Proof. Let $x^* \in \partial_2 f(z_n, x_n)$, It follows from condition (A4) and the definition of $\partial_2 f$ that

$$f(z_n, x) \geq f(z_n, x_n) + \langle x^*, x - x_n \rangle,$$

for all $x \in \Delta$. Taking the limits as $n \rightarrow \infty$, using the condition (A3*), we give

$$f(\bar{z}, x) \geq f(\bar{z}, \bar{x}) + \langle x^*, x - \bar{x} \rangle,$$

for all $x \in \Delta$. Hence, $x^* \in \partial_2 f(\bar{z}, \bar{x})$. □

Now, we prove the following proposition for Algorithm 2, which have important role in the proof of main result in this section.

Proposition 4.1. *For all $x^* \in E(f) \cap F(S)$ and all $n \in \mathbb{N}$, we get*

- (i) $\phi(x^*, w_n) \leq \phi(x^*, x_n) - (\frac{2}{\nu} - \frac{4}{c^2})\sigma_n^2 \|g_n\|^2$,
- (ii) $\phi(x^*, t_n) \leq \phi(x^*, x_n) - (1 - \alpha_n)(\frac{2}{\nu} - \frac{4}{c^2})\sigma_n^2 \|g_n\|^2$.

Proof. Using Lemma 2.7, the definition of V and inequality (2.8), we have

$$\begin{aligned} \phi(x^*, w_n) &= \phi(x^*, R_C J^{-1}(Jx_n - \sigma_n g_n)) \\ &\leq \phi(x^*, J^{-1}(Jx_n - \sigma_n g_n)) \\ &= V(x^*, Jx_n - \sigma_n g_n) \\ &\leq V(x^*, Jx_n - \sigma_n g_n + \sigma_n g_n) - 2\langle J^{-1}(Jx_n - \sigma_n g_n) - x^*, \sigma_n g_n \rangle \\ &= \phi(x^*, x_n) - 2\langle J^{-1}(Jx_n - \sigma_n g_n) - x^*, \sigma_n g_n \rangle \\ &= \phi(x^*, x_n) - 2\sigma_n \langle x_n - x^*, g_n \rangle + 2\langle J^{-1}(Jx_n - \sigma_n g_n) - x_n, -\sigma_n g_n \rangle. \end{aligned} \tag{4.9}$$

Since $g_n \in \partial_2 f(z_n, x_n)$, we get

$$\langle x_n - x^*, g_n \rangle \geq f(z_n, x_n) - f(z_n, x^*) \geq f(z_n, x_n) = \frac{\sigma_n \|g_n\|^2}{\nu}.$$

Therefore, we obtain

$$-\frac{2}{\nu}\sigma_n^2 \|g_n\|^2 \geq -2\sigma_n \langle x_n - x^*, g_n \rangle. \tag{4.10}$$

On the other hand, from Lemma 2.10, we get

$$\begin{aligned}
& 2\langle J^{-1}(Jx_n - \sigma_n g_n) - x_n, -\sigma_n g_n \rangle \\
&= 2\langle J^{-1}(Jx_n - \sigma_n g_n) - J^{-1}(Jx_n), -\sigma_n g_n \rangle \\
&\leq 2\|J(J^{-1}(Jx_n - \sigma_n g_n)) - J(J^{-1}Jx_n)\| \|\sigma_n g_n\| \\
&\leq \frac{4}{c^2} \sigma_n^2 \|g_n\|^2.
\end{aligned} \tag{4.11}$$

Where $\frac{1}{c}$ ($0 < c \leq 1$) is the 2-uniformly convex constant of E . Thus, combining inequalities (4.10) and (4.11), we can derive (i). A similar argument as in the proof of Lemma 3.3 shows that

$$\phi(x^*, t_n) \leq \alpha_n \phi(x^*, x_n) + (1 - \alpha_n) \phi(x^*, w_n).$$

Using (i), we see that

$$\begin{aligned}
\phi(x^*, t_n) &\leq \alpha_n \phi(x^*, x_n) + (1 - \alpha_n) \phi(x^*, x_n) - (1 - \alpha_n) \left(\frac{2}{\nu} - \frac{4}{c^2} \right) \sigma_n^2 \|g_n\|^2 \\
&= \phi(x^*, x_n) - (1 - \alpha_n) \left(\frac{2}{\nu} - \frac{4}{c^2} \right) \sigma_n^2 \|g_n\|^2.
\end{aligned}$$

Therefore (ii) is proved. \square

Theorem 4.1. *Assume that $S : C \rightarrow C$ is a relatively nonexpansive mapping such that*

$$\Omega := E(f) \cap F(S) \neq \emptyset.$$

Then the sequences $\{x_n\}_{n=1}^\infty$, $\{y_n\}_{n=1}^\infty$, $\{z_n\}_{n=1}^\infty$, $\{w_n\}_{n=1}^\infty$ and $\{t_n\}_{n=1}^\infty$ generated by Algorithm 2 converge strongly to the some solution $u^ \in \Omega$, where $u^* = R_\Omega x_0$ and R_Ω is the sunny generalized nonexpansive retraction from E onto Ω .*

Proof. Let $x^* \in \Omega$. Similar to the proof of Theorem 3.1, we have

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0 \quad \& \quad \lim_{n \rightarrow \infty} \|x_n - t_n\| = 0, \tag{4.12}$$

which imply that $\{x_n\}$ and consequently $\{t_n\}$ converge strongly to $\bar{x} \in C$ and

$$\lim_{n \rightarrow \infty} (\phi(x^*, x_n) - \phi(x^*, t_n)) = 0. \tag{4.13}$$

Since $(1 - \alpha_n) \left(\frac{2}{\nu} - \frac{4}{c^2} \right) > 0$, it follows from Lemma 4.1 (ii) that $\lim_{n \rightarrow \infty} \sigma_n \|g_n\| = 0$.

Now, we prove that $\{y_n\}$, $\{z_n\}$ and $\{g_n\}$ are bounded and $f(z_n, x_n) \rightarrow 0$, as $n \rightarrow \infty$. Suppose that

$$A(y) = \lambda_n f(x_n, y) + \frac{1}{2} \phi(y, x_n).$$

Since ϕ is lower semicontinuous respect to the first argument y , from Lemma 2.2, we deduce

$$\partial A(y) = \lambda_n \partial_2 f(x_n, y) + \frac{1}{2} \nabla_1 \phi(y, x_n).$$

Let $u_1, u_2 \in C$, $w_1 \in \partial_2 f(x_n, u_1)$ and $w_2 \in \partial_2 f(x_n, u_2)$, we have

$$\langle w_1, u_1 - y \rangle \geq f(x_n, u_1) - f(x_n, y), \quad \forall y \in C, \tag{4.14}$$

$$\langle -w_2, y - u_2 \rangle \geq f(x_n, u_2) - f(x_n, y), \quad \forall y \in C. \tag{4.15}$$

Set $y = u_2$ in inequality (4.14) and $y = u_1$ in inequality (4.15), we get

$$\langle w_1 - w_2, u_1 - u_2 \rangle \geq 0. \quad (4.16)$$

On the other hand, we have

$$\frac{1}{2} \nabla_1 \phi(u_i, x_n) = Ju_i - Jx_n, \quad i = 1, 2. \quad (4.17)$$

Therefore, using Lemma 2.11, there exists $\tau > 0$ such that

$$\langle Ju_1 - Ju_2, u_1 - u_2 \rangle \geq \tau \|u_1 - u_2\|^2. \quad (4.18)$$

From inequalities (4.16) and (4.18), we obtain

$$\langle (\lambda_n w_1 + Ju_1 - Jx_n) - (\lambda_n w_2 + Ju_2 - Jx_n), u_1 - u_2 \rangle \geq \tau \|u_1 - u_2\|^2. \quad (4.19)$$

For all $u \in C$, put $T_n(u) := \lambda_n w + Ju - Jx_n$, where $w \in \partial_2 f(x_n, u)$. So $T_n(u) \subseteq \partial A(u)$ for all $u \in C$. Therefore it follows from inequality (4.19) that

$$\langle t^n(u_1) - t^n(u_2), u_1 - u_2 \rangle \geq \tau \|u_1 - u_2\|,$$

for all $u_1, u_2 \in C$, all $t^n(u_1) \in T_n(u_1)$ and $t^n(u_2) \in T_n(u_2)$, this means T_n is multivalued monotone.

Using Lemma 2.1 and Lemma 2.2, we have

$$y_n = \min_{y \in C} \lambda_n f(x_n, y) + \frac{1}{2} \phi(y, x_n),$$

if only if

$$0 \in \partial A(y_n) + N_C(y_n).$$

Which implies that $-T_n(y_n) \subseteq N_C(y_n)$, thus

$$\langle t^n(y_n), y - y_n \rangle \geq 0, \quad (4.20)$$

for all $y \in C$ and all $t^n(y_n) \in T_n(y_n)$. Replacing u_1 and u_2 by x_n and y_n in inequality (4.19), respectively and interchanging y by x_n in inequality (4.20), we have

$$\langle t^n(x_n), x_n - y_n \rangle \geq \langle t^n(y_n), x_n - y_n \rangle + \tau \|x_n - y_n\|^2 \geq \tau \|x_n - y_n\|^2, \quad (4.21)$$

for all $t^n(x_n) \in T_n(x_n)$ and all $t^n(y_n) \in T_n(y_n)$. So, $t^n(x_n) \in \partial A(x_n)$, for all $t^n(x_n) \in T_n(x_n)$ and since $\frac{1}{2} \nabla_1 \phi(x_n, x_n) = 0$, we obtain $t^n(x_n) \in \lambda_n \partial_2 f(x_n, x_n)$. Since $x_n \rightarrow \bar{x}$, it follows from Lemma 4.3 that

$$\partial_2 f(x_n, x_n) \subset \partial_2 f(\bar{x}, \bar{x}).$$

Thus, we can deduce from Lemma 2.3 that $t^n(x_n)$ is bounded, because of $0 < \lambda \leq \lambda_n \leq 1$. So, from inequality (4.21), we obtain $\tau \|x_n - y_n\| \leq \|t^n(x_n)\|$. Therefore, we can conclude that $\{y_n\}$ is bounded. Since $\{z_n\}$ is a convex combination of $\{x_n\}$ and $\{y_n\}$, it is also bounded, hence, there exists a subsequence of $\{z_n\}$, again denoted by $\{z_n\}$, which converges weakly to $\bar{z} \in C$. In a similar way, it follows from lemmas 2.3 and 4.3 that the sequence $\{g_n\}$ is bounded.

If $x_n = y_n$ then we have $f(z_n, x_n) = 0$ and $\sigma_n = 0$ and if $x_n \neq y_n$, by the definition of σ_n , we obtain

$$\nu f(z_n, x_n) = \sigma_n \|g_n\| \|g_n\| \rightarrow 0 \implies f(z_n, x_n) \rightarrow 0,$$

since $\sigma_n \|g_n\| \rightarrow 0$ and $0 < \nu < \frac{c^2}{2}$.

Now, we show that $\bar{x} \in E(f)$ and $\|x_n - y_n\| \rightarrow 0$. If $y_n = x_n$ then it follows from Lemma 3.1 (i), that

$$\lambda_n f(x_n, y) \geq 0, \quad (4.22)$$

for all $y \in C$. By letting $n \rightarrow \infty$ in inequality (4.22) and using condition (A3*), we get $f(\bar{x}, y) \geq 0$, because of $0 < \lambda \leq \lambda_n \leq 1$, i.e., $\bar{x} \in E(f)$.

Now, we let that $y_n \neq x_n$, since $f(z_n, \cdot)$ is convex, we obtain

$$\begin{aligned} \rho_n f(z_n, y_n) + (1 - \rho_n) f(z_n, x_n) \\ \geq f(z_n, \rho_n y_n + (1 - \rho_n) x_n) = f(z_n, z_n) = 0. \end{aligned}$$

Therefore,

$$\rho_n [f(z_n, x_n) - f(z_n, y_n)] \leq f(z_n, x_n) \rightarrow 0, \quad (4.23)$$

as $n \rightarrow \infty$. By the Step 3.1 of Algorithm 2 and inequality (4.23), we have

$$\frac{\alpha \rho_n}{2\lambda_n} \phi(y_n, x_n) \leq \rho_n [f(z_n, x_n) - f(z_n, y_n)] \leq f(z_n, x_n) \rightarrow 0. \quad (4.24)$$

Now, we consider two cases:

Case 1: $\limsup_{n \rightarrow \infty} \rho_n > 0$.

In this case, there exists $\bar{\rho} > 0$ and a subsequence of $\{\rho_n\}$, again denoted by $\{\rho_n\}$, such that $\rho_n \rightarrow \bar{\rho}$ and since $0 < \lambda \leq \lambda_n \leq 1$, from inequality (4.24), we can conclude that

$$\phi(y_n, x_n) \rightarrow 0.$$

Thus, from Lemma 2.14, we have $\|y_n - x_n\| \rightarrow 0$, which implies $y_n \rightarrow \bar{x}$.

Case 2: $\lim_n \rho_n \rightarrow 0$.

Let m be the smallest nonnegative integer such that the Step 3.1 of Algorithm 2 is satisfied, i.e.,

$$f(z_{n,m}, x_n) - f(z_{n,m}, y_n) \geq \frac{\alpha}{2\lambda} \phi(y_n, x_n),$$

where $z_{n,m} = (1 - \gamma^m)x_n + \gamma^m y_n$. So,

$$f(z_{n,m-1}, x_n) - f(z_{n,m-1}, y_n) < \frac{\alpha}{2\lambda_n} \phi(y_n, x_n). \quad (4.25)$$

On the other hand, setting $y = x_n$ in Lemma 3.1 (i), condition (A1) and equality (2.6) imply that

$$-\lambda_n f(x_n, y_n) \geq \langle Jy_n - Jx_n, y_n - x_n \rangle = \frac{1}{2} \phi(y_n, x_n) + \frac{1}{2} \phi(x_n, y_n).$$

Therefore,

$$\frac{1}{2} \phi(y_n, x_n) \leq -\lambda_n f(x_n, y_n). \quad (4.26)$$

From inequalities (4.25) and (4.26), we have

$$f(z_{n,m-1}, x_n) - f(z_{n,m-1}, y_n) < -\alpha f(x_n, y_n). \quad (4.27)$$

Taking the limits as $n \rightarrow \infty$ in above inequality, we obtain $z_{n,m-1} \rightarrow \bar{x}$, since $\gamma^m = \rho_n \rightarrow 0$. Because of $\{y_n\}$ is bounded, there exists a subsequence of $\{y_n\}$, again denoted by $\{y_n\}$, which converges weakly to $\bar{y} \in C$. By letting $n \rightarrow \infty$ in inequality (4.27) and using conditions (A1) and (A3*), we get

$$-f(\bar{x}, \bar{y}) \leq -\alpha f(\bar{x}, \bar{y}). \quad (4.28)$$

Which implies that $f(\bar{x}, \bar{y}) \geq 0$, because of $\alpha \in (0, 1)$. So, If we take the limits as $n \rightarrow \infty$ in inequality (4.26), then we can conclude that $\phi(y_n, x_n) \rightarrow 0$. Thus, from Lemma 2.14, we have $\|y_n - x_n\| \rightarrow 0$, which implies that $y_n \rightarrow \bar{x}$. By Lemma 3.1, we have

$$\lambda_n[f(x_n, y) - f(x_n, y_n)] \geq \langle Jy_n - Jx_n, y_n - y \rangle, \quad (4.29)$$

for all $y \in C$. By letting $n \rightarrow \infty$ in inequality (4.29), it follows that $f(\bar{x}, y) \geq 0$, because of $0 < \lambda \leq \lambda_n \leq 1$, this means $\bar{x} \in E(f)$.

Now, we show that $\bar{x} \in F(S)$. Let $r_1 = \sup_{n \geq 0} \{\|x_n\|, \|w_n\|\}$ and $r_2 = \sup_{n \geq 0} \{\|w_n\|, \|Sw_n\|\}$. Using Lemma 2.13, there exists a continuous, strictly increasing and convex function $g_1 : [0, 2r_1] \rightarrow \mathbb{R}$ with $g_1(0) = 0$ such that for $x^* \in \Omega$, we get

$$\begin{aligned} \phi(x^*, t_n) &= \phi(x^*, J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)(\beta_n Jw_n + (1 - \beta_n)JSw_n))) \\ &\leq \|x^*\|^2 + \alpha_n \|x_n\|^2 + (1 - \alpha_n)\beta_n \|w_n\|^2 + (1 - \alpha_n)(1 - \beta_n) \|Sw_n\|^2 \\ &\quad - 2\alpha_n \langle x^*, Jx_n \rangle - 2(1 - \alpha_n)\beta_n \langle x^*, Jw_n \rangle - 2(1 - \alpha_n)(1 - \beta_n) \langle x^*, JSw_n \rangle \\ &\quad - \alpha_n(1 - \alpha_n)\beta_n g_1(\|Jw_n - Jx_n\|) \\ &\leq \phi(x^*, x_n) - \alpha_n(1 - \alpha_n)\beta_n g_1(\|Jw_n - Jx_n\|), \end{aligned} \quad (4.30)$$

and in a similar way, there exists a continuous, strictly increasing and convex function $g_2 : [0, 2r_2] \rightarrow \mathbb{R}$ with $g_2(0) = 0$ such that for $x^* \in \Omega$, we obtain

$$\phi(x^*, t_n) \leq \phi(x^*, x_n) - (1 - \alpha_n)^2 \beta_n (1 - \beta_n) g_2(\|Jw_n - JSw_n\|) = 0. \quad (4.31)$$

It follows from inequalities (4.30) and (4.31) that

$$\alpha_n(1 - \alpha_n)\beta_n g_1(\|Jw_n - Jx_n\|) \leq \phi(x^*, x_n) - \phi(x^*, t_n), \quad (4.32)$$

$$(1 - \alpha_n)^2 \beta_n (1 - \beta_n) g_2(\|Jw_n - JSw_n\|) \leq \phi(x^*, x_n) - \phi(x^*, t_n). \quad (4.33)$$

Taking the limits as $n \rightarrow \infty$ in inequalities (4.32) and (4.33), using Lemma 2.15 and equality (4.13), we obtain

$$\lim_{n \rightarrow \infty} g_1(\|Jw_n - Jx_n\|) = 0 \quad \& \quad \lim_{n \rightarrow \infty} g_2(\|Jw_n - JSw_n\|) = 0.$$

From the properties of g_1 and g_2 , we have

$$\lim_{n \rightarrow \infty} \|Jw_n - Jx_n\| = 0 \quad \& \quad \lim_{n \rightarrow \infty} \|Jw_n - JSw_n\| = 0.$$

Since J^{-1} is uniformly norm-to-norm continuous on bounded sets, we obtain

$$\lim_{n \rightarrow \infty} \|w_n - x_n\| = 0 \quad \& \quad \lim_{n \rightarrow \infty} \|w_n - Sw_n\| = 0. \quad (4.34)$$

So, we get $\bar{x} \in \hat{F}(S)$, because of $w_n \rightharpoonup \bar{x}$, therefore using the definition of S , we have that $\bar{x} \in F(S)$.

Setting $z = u^*$ in Lemma 2.7, since $x_{n+1} = R_{C_n \cap D_n} x_0$ and ϕ is continuous respect to the first argument, we obtain

$$\phi(u^*, x_0) \geq \lim_{n \rightarrow \infty} \phi(x_{n+1}, x_0) = \lim_{n \rightarrow \infty} \phi(x_n, x_0) = \phi(\bar{x}, x_0).$$

Also, since $u^* = R_\Omega x_0$, it follows from Lemma 2.8 that

$$\phi(u^*, x_0) \leq \phi(y, x_0),$$

for all $y \in \Omega$. Therefore $\bar{x} = u^*$ and consequently sequences $\{x_n\}_{n=1}^\infty$, $\{y_n\}_{n=1}^\infty$, $\{z_n\}_{n=1}^\infty$, $\{w_n\}_{n=1}^\infty$ and $\{t_n\}_{n=1}^\infty$ converge strongly to $R_\Omega x_0$. \square

5. NUMERICAL EXAMPLE

Now, we demonstrate theorems 3.1 and 4.1 with an example. Also, we compare the behavior of the sequence $\{x_n\}$ generated by algorithms 1 and 2.

Example 5.1. Let $E = \mathbb{R}$ and $C = [-100, 100]$. Define $f(x, y) := y^2 - 4x^2 + 3xy$.

We see that f satisfies the conditions (A1) – (A5) as follows:

(A1) $f(x, x) := x^2 - 4x^2 + 3x^2 = 0$ for all $x \in C$,

(A2) If $f(x, y) = (y - x)(y + 4x) \geq 0$, then

$$f(y, x) = (x - y)(x + 4y) = (x - y)((y + 4x) - 3(x - y)) = -f(x, y) - 3(x - y)^2 \leq 0,$$

for all $x, y \in C$, i.e., f is pseudomonotone, while f is not monotone.

(A3) If $x_n \rightharpoonup \bar{x}$ and $y_n \rightharpoonup \bar{y}$, then

$$f(x_n, y_n) = y_n^2 - 4x_n^2 + 3x_n y_n \rightarrow \bar{y}^2 - 4\bar{x}^2 + 3\bar{x}\bar{y} = f(\bar{x}, \bar{y}),$$

i.e., f is jointly weakly continuous on $C \times C$.

(A4) Let $\theta \in (0, 1)$. Since

$$\begin{aligned} f(x, \theta y_1 + (1 - \theta)y_2) &= (\theta y_1 + (1 - \theta)y_2)^2 - 4x^2 + 3x(\theta y_1 + (1 - \theta)y_2) \\ &\leq \theta(y_1^2 - 4x^2 + 3xy_1) + (1 - \theta)(y_2^2 - 4x^2 + 3xy_2) \\ &= \theta f(x, y_1) + (1 - \theta)f(x, y_2), \end{aligned}$$

so $f(x, \cdot)$ is convex, also, $\liminf_{y \rightarrow \bar{y}} f(x, y) = f(x, \bar{y})$, hence $f(x, \cdot)$ is lower semicontinuous. Since $\partial_2 f(x, y) = 2y + 3x$, thus $f(x, \cdot)$ is subdifferentiable on C for each $x \in C$.

(A5) Since $\phi(y, x) = (y - x)^2$, we get

$$\begin{aligned} f(x, y) + f(y, z) &= z^2 - 4x^2 - 3y^2 + 3xy + 3zy \\ &= f(x, z) - \frac{3}{2}(y - x)^2 - \frac{3}{2}(z - y)^2 + \frac{3}{2}(x - z)^2 \\ &\geq f(x, z) - \frac{3}{2}(y - x)^2 - \frac{3}{2}(z - y)^2, \end{aligned} \tag{5.1}$$

i.e., f satisfies the ϕ -Lipschitz-type condition with $c_1, c_2 = \frac{3}{2}$.

Now, define $S : C \rightarrow C$ by $Sx = \frac{x}{5}$ for all $x \in C$, so $F(S) = \{0\}$ and

$$\begin{aligned}\phi(0, Sx) &= \phi(0, \frac{x}{5}) \\ &= 0 - 2\langle 0, \frac{x}{5} \rangle + |\frac{x}{5}|^2 \\ &\leq |x|^2 \\ &= \phi(0, x),\end{aligned}\tag{5.2}$$

for all $x \in C$. Let $x_n \rightarrow p$ such that $\lim_{n \rightarrow \infty} (Sx_n - x_n) = 0$, this implies that $\hat{F}(S) = \{0\}$.

Thus, $\hat{F}(S) = F(S)$, i.e., S is relatively nonexpansive mapping. On the other hand, if for

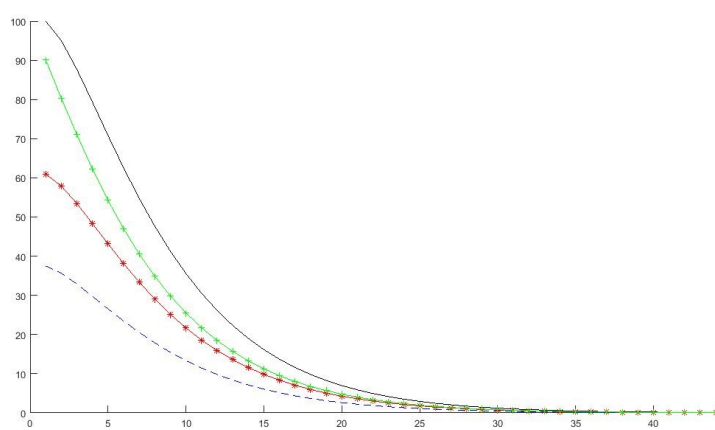


FIGURE 1. Extragradient Algorithm

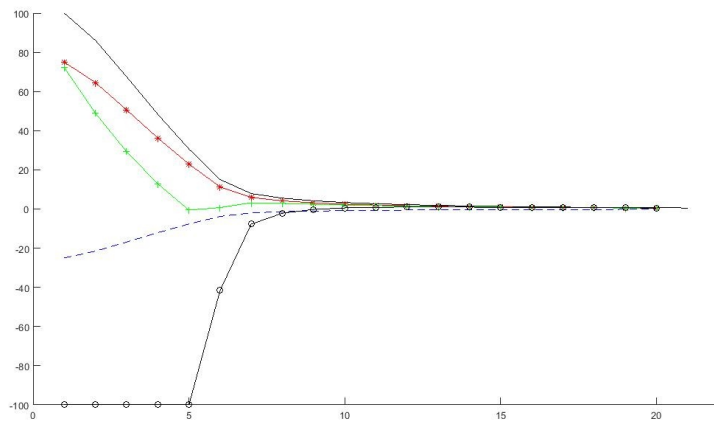


FIGURE 2. Linesearch Algorithm

each $y \in C$, $f(x, y) \geq 0$, then $x = 0$, i.e., $E(f) = \{0\}$ and consequently $\Omega = E(f) \cap F(S) = \{0\}$. Also, assume that $\alpha_n = \frac{1}{2} + \frac{1}{3+n}$, $\beta_n = \frac{1}{3} + \frac{1}{4+n}$ for all $n \geq 0$ and $x_0 = 100$.

In Extragradient Algorithm, if $\lambda_n = \frac{1}{6}$, then we have

$$\frac{1}{6}f(x_n, y_n) + \frac{1}{2}(y_n - x_n)^2 = \min_{y \in C} \left\{ \frac{1}{6}f(x_n, y) + \frac{1}{2}(y - x_n)^2 \right\},$$

i.e., $y_n = \frac{3}{8}x_n$, also

$$\frac{1}{6}f(y_n, z_n) + \frac{1}{2}(z_n - x_n)^2 = \min_{y \in C} \left\{ \frac{1}{6}f(y_n, y) + \frac{1}{2}(y - x_n)^2 \right\},$$

i.e., $z_n = \frac{39}{24}y_n = \frac{39}{64}x_n$, therefore

$$t_n = \alpha_n x_n + (1 - \alpha_n) \left[\beta_n z_n + \frac{1}{5}(1 - \beta_n)z_n \right],$$

and $x_{n+1} = R_{C_n \cap D_n} x_0$ or $|x_{n+1} - x_0| = \min_{z \in C_n \cap D_n} |z - x_0|$, where

$$\begin{cases} C_n = \{z \in C : |t_n - z| \leq |x_n - z|\}, \\ D_n = \{z \in C : (x_n - z)(x_0 - x_n) \geq 0\}. \end{cases}$$

Table1 Numerical Results for Algorithm 1				
n	x_n	y_n	z_n	t_n
0	100	37.5	60.94	90.104
1	95.05	35.65	57.92	80.36
2	87.71	32.89	53.47	71.016
3	79.36	29.61	48.36	62.277
	\vdots	\vdots	\vdots	\vdots
37	0.2826	0.1059	0.1722	0.1881
38	0.2353	0.0882	0.1434	0.1565
39	0.1959	0.0734	0.1194	0.1302
40	0.163	0.0611	0.0993	0.1082
	\vdots	\vdots	\vdots	\vdots
70	0.0003	0.000113	0.000183	0.000197
71	0.0002	$7.50e - 05$	0.000122	0.000131
72	0.0001	$3.75e - 05$	$6.09e - 05$	$6.55e - 05$
73	0	0	0	0

Table2 Numerical Results for Algorithm 2

n	x_n	y_n	z_n	t_n	w_n
0	100	-25	75	72.22	-100
1	86.11	-21.53	64.53	48.91	-100
2	67.51	-16.88	50.63	29.26	-100
3	48.38	-12.097	36.29	12.89	-100
	\vdots	\vdots	\vdots	\vdots	\vdots
37	0.0561	-0.01403	0.0421	0.0422	0.0553
38	0.0491	-0.01228	0.0368	0.0369	0.0485
39	0.043	-0.01075	0.0322	0.0329	0.0426
40	0.0376	-0.0094	0.0282	0.0283	0.0373
	\vdots	\vdots	\vdots	\vdots	\vdots
70	0.0003	-7.50e-05	0.00022	0.00022	0.0003
71	0.0002	-5.00e-05	0.00015	0.00015	0.0002
72	0.0001	-2.50e-05	7.50e-5	7.46e-5	1.00e-04
73	0	0	0	0	0

Since $\Omega = \{0\}$, we get $R_\Omega(x_0) = 0$. Moreover, numerical results for Algorithm 1 show that the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{t_n\}$ converge strongly to 0.

In Linesearch Algorithm, x_n is the same in Extragradient Algorithm. Assume that $\lambda_n = \frac{1}{2}$, $\alpha = \frac{1}{2}$, $\gamma = 0.2$, $\nu = \frac{1}{4}$ and $c = 1$. So,

$$\frac{1}{2}f(x_n, y_n) + \frac{1}{2}(y_n - x_n)^2 = \min_{y \in C} \frac{1}{2}\{f(x_n, y) + (y - x_n)^2\},$$

i.e., $y_n = -\frac{1}{4}x_n$, and m is the smallest nonnegative integer such that

$$(x_n - y_n)\left(\frac{1}{2}x_n + \frac{3}{2}y_n + 3z_n\right) \geq 0,$$

where

$$z_n = z_{n,m} = (1 - (0.2)^m)x_n + (0.2)^m y_n.$$

Also, $g_n = 2x_n + 3z_n$ and $|w_n - (J^{-1}(Jx_n - \sigma_n g_n))| = \min_{z \in C} |z - (J^{-1}(Jx_n - \sigma_n g_n))|$. Since $y_n \neq x_n$, then

$$\sigma_n = \frac{\frac{1}{4}(x_n^2 - 4z_n^2 + 3x_n z_n)}{|g_n|},$$

and

$$t_n = \alpha_n x_n + (1 - \alpha_n)[\beta_n w_n + \frac{1}{5}(1 - \beta_n)w_n].$$

Furthermore, numerical results for Algorithm 2 show that the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, $\{t_n\}$ and $\{w_n\}$ converge strongly to 0. By comparing Figure1 and Figure2, we see that the speed of convergence of the sequence $\{x_n\}$ generated by Linesearch Algorithm is equal to Extragradient Algorithm. The computations associated with example were performed using MATLAB (Step: 10^{-4}) software.

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